数理解析研究所講究録
1519 巻 2006 年 167-174

A note on invariant Hilbert spaces of holomorphic functions on the unit ball in $\mathbb{C}^d$

Penghui Wang

1 Introduction

Invariant Hilbert spaces of holomorphic functions on bounded symmetric domains have been extensively studied[Ara]. The study is motivated by the unitary representation of the automorphism group of the bounded symmetric domains.

Let Ω be a bounded symmetric domain, and $\mathrm{Aut}(\Omega)$ denote the automorphism group of Ω. Let G denote the connected component of the identity in $\mathrm{Aut}(\Omega)$. Then G can be naturally represented on the Bergman space $L_a^2(\Omega)$, the representation map π is defined by

$$\pi(\varphi)f = f\circ\varphi\cdot J\varphi,\ f\in L_a^2(\Omega),\ \varphi\in G,$$

where $J\varphi$ is the complex Jacobian of φ. Moreover, this representation is unitary, that is, for any $\varphi\in G$, the operator $\pi(\varphi)$ is unitary. For natural Hilbert space H of holomorphic functions on Ω, the similar action of G on H can also been defined. J. Arazy[Ara] shows that, with some mild assumptions, the only Hilbert space which makes π be a unitary representation is the Bergman space. Of cause, J. Arazy deals with a more complicated case. For detailed information, one can refer to [Ara].

In this note, we will mainly concern Hilbert spaces of holomorphic functions on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$. In this case, the automorphism group $\mathrm{Aut}(\mathbb{B}_d)$

---

[0]Supported by Specialized Research Fund for the Doctoral Program of Higher Education.

[1]2000 AMS Subject Classification: 47A13; 47A20; 46H25; 46C99.

can be written precisely. In fact, by [Ru, Theorem 2.2.5], $\mathrm{Aut}(\mathbb{B}_d)$ is generated by the unitary group $\mathcal{U}_d$ of $\mathbb{C}^d$ and $\{\varphi_\lambda |\, \lambda \in \mathbb{B}_d\}$, where, for any $\lambda \in \mathbb{B}_d$, φ_λ is defined as follows. If $\lambda = 0$, $\varphi_\lambda(z) = -z$. If $\lambda \neq 0$,

$$\varphi_\lambda = \frac{\lambda - P_\lambda z - \sqrt{1-|\lambda|^2} P_\lambda^\perp z}{1 - \langle z, a \rangle}, \tag{1.1}$$

where P_λ is the orthogonal projection from $\mathbb{C}^d$ onto the complex line $[\lambda]$ spanned in $\mathbb{C}^d$ by λ, and $P_\lambda^\perp = I - P_\lambda$. Therefore, one can only consider the automorphism with the expression (1.1). We rewrite the above representation $\pi(\varphi_\lambda)$ as U_λ in short, that is

$$U_\lambda f = f \circ \varphi_\lambda \cdot J\varphi_\lambda.$$

After some calculation, it is not difficult to see that the complex Jacobian $J\varphi_\lambda = (-1)^d \frac{(1-|\lambda|^2)^{\frac{d+1}{2}}}{(1-\langle z, \lambda\rangle)^{d+1}}$ is just the normalized Bergman kernel on $\mathbb{B}_d$ multiplied by $(-1)^d$.

For many interesting unitary invariant reproducing Hilbert space H on $\mathbb{B}_d$, one can define the similar action by $V_\lambda f = f \circ \varphi_\lambda \cdot k_\lambda$, where k_λ is the normalized reproducing kernel of H. So, the question is, when V_λ is unitary? In other word, to ensure that V_λ is unitary, the complex Jacobian $J\varphi_\lambda$ can be replaced to what kind of 'good' functions.

In this note, with some mild assumptions, we will prove that if V_λ is unitary, then there is a positive number μ, such that $k_\lambda = ((-1)^d J\varphi_\lambda)^\mu$.

We organize this note as follows. In section 2, we will introduce some notations of unitary invariant reproducing kernel. In section 3, we prove the main theorem.

2 Preliminaries

From a general theory of reproducing kernels [Aro], one sees that a reproducing function space is uniquely determined by its kernel. In this paper, we will mainly concern unitary invariant reproducing function space of holomorphic functions on $\mathbb{B}_d$. A reproducing function space is called unitary invariant, if for any unitary operator U on $\mathbb{C}^d$, $f \circ U \in H$ whenever $f \in H$, and for all $f, g \in H$,

$$\langle f \circ U, g \circ U \rangle = \langle f, g \rangle.$$

By [GHX], H is unitary invariant if and only if for any unitary operator U on $\mathbb{C}^d$

$$K_{U\lambda}(Uz) = K_\lambda(z);$$

and this holds if and only if there is a holomorphic function on the unit disk $f(z) = \sum_{n=1}^{\infty} a_n z^n$ with $a_n \geq 0$, such that

$$K_\lambda(z) = f(\langle z, \lambda\rangle).$$

Without loss of generality, we will consider the case that all the $a_n > 0$, and $a_0 = 1$. Hence, by [GHX, Proposition 4.1], H has a canonical orthonormal basis $\{[a_{|\alpha|}\frac{|\alpha|!}{\alpha!}]^{1/2}z^\alpha\}$, and $\|z^\alpha\| = [\frac{\alpha!}{a_{|\alpha|}|\alpha|!}]^{\frac{1}{2}}$. Particularly, $\|1\| = 1$.

Example. Let $H^2_\mu(\mathbb{B}_d)$ be the reproducing function space defined by the reproducing kernel $K^{(\mu)}_\lambda = \frac{1}{(1-\langle z,\lambda\rangle)^\mu}$ $(\mu > 0)$. It is easy to verify that $H^2_\mu(\mathbb{B}_d)$ is unitary invariant. When $\mu = 1$, $H^2_\mu(\mathbb{B}_d)$ is the symmetric Fock space H^2_d, which is deeply studied by W. Arveson[Arv]. When $\mu = d$, $H^2_\mu(\mathbb{B}_d)$ is the Hardy space $H^2(\mathbb{B}_d)$. When $\mu > d$, $H^2_\mu(\mathbb{B}_d)$ is the weighted Bergman space $L^2_a[(1-|z|^2)^{\mu-d-1}dV]$, and in particular $H^2_{d+1}(\mathbb{B}_d)$ is the usual Bergman space.

By [Guo, Section 4], for a given $\mu > 0$, the operator

$$V_\lambda f = f \circ \varphi_\lambda \cdot \frac{(1-|\lambda|^2)^{\frac{\mu}{2}}}{(1-\langle \cdot, \lambda\rangle)^\mu}$$

is a unitary operator on $H^2_\mu(\mathbb{B}_d)$ (For the case $\mu = 1$, this is also proved by D. Greene[Gr, Theorem 3.3]). Notice that $\frac{(1-|\lambda|^2)^{\frac{\mu}{2}}}{(1-\langle\cdot,\lambda\rangle)^\mu}$ is the normalized reproducing kernel of $H^2_\mu(\mathbb{B}_d)$.

3 The proof of the main theorem

In this section, we will prove the main theorem. As in Section 2, let H be a unitary invariant reproducing functions space with the reproducing kernel K_λ. For any $\lambda \in \mathbb{B}_d$, define an operator V_λ on H by $V_\lambda f = f \circ \varphi_\lambda \cdot k_\lambda$, where k_λ is the normalized reproducing kernel. We have the following theorem.

Theorem 3.1. *With the above notations, if V_λ is a unitary operator on H, then there is a positive number μ such that,*

$$k_\lambda = \frac{(1-|\lambda|^2)^{\frac{\mu}{2}}}{(1-\langle\cdot,\lambda\rangle)^\mu}.$$

Proof. Below, we will prove that if V_λ is unitary, then the reproducing kernel $K_\lambda = \sum\limits_{n=0}^{\infty} a_n \langle z, \lambda\rangle^n$ is uniquely determined by a_1, that is,

Claim. For $n > 1$, each a_n can be uniquely expressed by a_1.

We will prove the claim by induction.

At first, we will calculate a_2. Taking $\lambda = (r, 0, \cdots, 0)$, we simply write $\varphi_\lambda = \varphi_r$ and $k_\lambda = k_r$. Since $z_1 = z_1 \circ \varphi_r \circ \varphi_r$, we have

$$\|z_1 k_r\|^2 = \|z_1 \circ \varphi_r\|^2 \tag{3.1}$$

We first calculate the left side of (1). By [GHX, Proposition 4.1], $\|z_1^n\|^2 = \frac{1}{a_n}$, and $\langle z_1^n, z_1^m\rangle = 0$ whenever $n \neq m$.

$$\|z_1 k_r(z)\|^2 = \frac{\|\sum\limits_{n=0}^{\infty} a_n r^n z_1^{n+1}\|^2}{\sum\limits_{n=0}^{\infty} a_n r^{2n}} = \frac{\sum\limits_{n=0}^{\infty} a_n^2 r^{2n} \|z_1^{n+1}\|^2}{\sum\limits_{n=0}^{\infty} a_n r^{2n}} = \frac{\sum\limits_{n=0}^{\infty} \frac{a_n^2}{a_{n+1}} r^{2n}}{\sum\limits_{n=0}^{\infty} a_n r^{2n}}.$$

And now we calculate the right side of (3.1),

$$\begin{aligned}
\|z_1 \circ \varphi_r\|^2 &= \|(r - z_1)\sum_{n=0}^{\infty}(rz_1)^n\|^2 \\
&= \|\sum_{n=0}^{\infty}(r^{n+1}z_1^n - r^n z_1^{n+1})\|^2 \\
&= \|r + \sum_{n=1}^{\infty}(r^{n+1} - r^{n-1})z_1^n\|^2 \\
&= r^2 + \sum_{n=1}^{\infty} \frac{r^{2n-2}(r^4 - 2r^2 + 1)}{a_n}.
\end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \frac{a_n^2}{a_{n+1}} r^{2n} = (\sum_{m=0}^{\infty} a_m r^{2m})(r^2 + \sum_{n=1}^{\infty} \frac{r^{2n-2}(r^4-2r^2+1)}{a_n}). \tag{3.2}$$

Comparing the coefficients of r^2 in both sides of (3.2) first, we have

$$\frac{a_1^2}{a_2} = 1 - \frac{2}{a_1} + \frac{1}{a_2} + \frac{a_1}{a_1}.$$

Therefore, when $a_1 \neq 1$,

$$a_2 = \frac{a_1(a_1+1)}{2}. \tag{3.3}$$

When $a_1 = 1$, to determine a_2, we compare the coefficient of r^4 in both sides of (3.2). After some simple computation, we have

$$\frac{a_2^2}{a_3} = \frac{1}{a_3} - \frac{1}{a_2} + a_2. \tag{3.4}$$

We also need the following equation.

$$\|z_1^2 \circ \varphi_r \cdot k_r\|^2 = \|z_1^2\|^2 = \frac{1}{a_2}.$$

Thus,

$$\|z_1^2 \circ \varphi_r \cdot K_r\|^2 = \frac{1}{a_2} \sum_{n=0}^{\infty} a_n r^{2n}. \tag{3.5}$$

Now, let us calculate the left side of (3.5). A careful verification shows that

$$\begin{aligned}
&\|z_1^2 \circ \varphi_r \cdot K_r\|^2 = \|(\frac{r - z_1}{1 - r z_1})^2 K_r\|^2 \\
= \; & \|(r - z_1)^2[\sum_{n=0}^{\infty}(n+1)(rz_1)^n][\sum_{m=0}^{\infty} a_m (rz_1)^m]\|^2 \\
= \; & \|r^2 + (r^2(2r + a_1 r) - 2r)z_1 \\
& + \sum_{n=2}^{\infty} r^{n-2}(r^4 \sum_{j=1}^{n+1} j a_{n+1-j} - 2r^2 \sum_{j=1}^{n} j a_{n-j} + \sum_{j=1}^{n-1} j a_{n-1-j}) z_1^n\|^2
\end{aligned}$$

Now, set $b_n = \sum\limits_{j=1}^{n-1} ja_{n-1-j}$, and the above equation can be simplified as follows.

$$\begin{aligned}
&\|z_1^2 \circ \varphi_r \cdot K_r\|^2 \\
= \;& \|r^2 + (r^2(2r + a_1 r) - 2r)z_1 + \sum_{n=2}^{\infty} r^{n-2}(r^4 b_{n+2} - 2r^2 b_{n+1} + b_n)z_1^n\|^2 \\
= \;& r^4 + [r^2(2r + a_1 r) - 2r]^2 \frac{1}{a_1} + \sum_{n=2}^{\infty} [r^{n-2}(r^4 b_{n+2} - 2r^2 b_{n+1} + b_n)]^2 \frac{1}{a_n} \\
= \;& r^4 + [r^3(2 + a_1) - 2r]^2 \frac{1}{a_1} + \sum_{n=2}^{\infty} r^{2n-4}[r^8 b_{n+2}^2 \\
& -4r^6 b_{n+2}b_{n+1} + r^4(4b_{n+1}^2 + 2b_{n+2}b_n) - 4r^2 b_{n+1}b_n + b_n^2]\frac{1}{a_n} \\
= \;& \frac{b_2^2}{a_2} + r^2(\frac{4}{a_1} - \frac{4b_3 b_2}{a_2} + \frac{b_3^2}{a_3}) \\
& + \sum_{n=2}^{\infty} r^{2n}[\frac{b_{n+2}^2}{a_{n+2}} + C(a_1, \cdots, a_{n+1}, b_2, \cdots, b_{n+2})],
\end{aligned}$$

where $C(a_1, \cdots, a_{n+1}, b_1, \cdots, b_{n+2})$ can be uniquely expressed by $\{a_i\}_{i=1}^{n+1}$ and $\{b_i\}_{i=2}^{n+2}$. Now comparing the coefficients of r^2 in both sides of (3.5), we have

$$\tfrac{4}{a_1} - \tfrac{2\cdot 2(2+a_1)}{a_2} + \tfrac{(2+a_1)^2}{a_3} = \tfrac{1}{a_2}. \tag{3.6}$$

When $a_1 = 1$, combining (3.4) with (3.6), we have

$$a_2 = 1 = \tfrac{a_1(a_1+1)}{2}$$

Hence, by (3.3) and (3.7), the equality $a_2 = \frac{a_1(a_1+1)}{2}$ is always true.

And now we assume that a_j is uniquely expressed by a_1 for $1 < j \leq m$. To prove a_{m+1} is uniquely expressed by a_1, we compare the coefficient of $r^{2(m-1)}$ in both sides of (3.5).

$$\frac{a_{m-1}}{a_2} = \frac{b_{m+1}^2}{a_{m+1}} + C(a_1, \cdots, a_m, b_2, \cdots, b_{m+1}).$$

By the definition of b_i, we know that b_i is uniquely expressed by $\{a_j\}_{j=1}^{i-2}$. By the inductive assumption, both a_{m-1} and $C(a_1, \cdots, a_m, b_2, \cdots, b_{m+1})$ are uniquely expressed by a_1, and so is a_{m+1}. Thus the claim is proved.

Set $\mu = a_1$. By section 2, if

$$K_\lambda(z) = \frac{1}{(1-\langle z, \lambda\rangle)^\mu} = 1 + \mu\langle z, \lambda\rangle + \sum_{n=2}^{\infty} \frac{\mu(\mu+1)\cdots(\mu+n-1)}{n!}\langle z, \lambda\rangle^n,$$

then V_λ is unitary. The above reasoning thus shows that

$$a_n = \frac{\mu(\mu+1)\cdots(\mu+n-1)}{n!}.$$

This means $K_\lambda(z) = \frac{1}{(1-\langle z, \lambda\rangle)^\mu}$, which implies that $k_\lambda = \frac{(1-|\lambda|^2)^{\frac{\mu}{2}}}{(1-\langle \cdot, \lambda\rangle)^\mu}$. □

Proposition 3.2. *Let H and H' be two unitary invariant reproducing function spaces on $\mathbb{B}_d$ with the reproducing kernels K_λ and K'_λ relatively. If*

$$\|f \circ \varphi_\lambda \cdot k'_\lambda\| = \|f\| \quad for\ \forall f \in H,$$

then $H = H'$, and hence by Theorem 3.1 $H = H^2_\mu(\mathbb{B}_d)$ for some $\mu > 0$.

Proof. Write $K_\lambda(z) = \sum_{n=0}^{\infty} a_n \langle z, \lambda\rangle^n$ and $K'_\lambda(z) = \sum_{n=0}^{\infty} b_n\langle z, \lambda\rangle^n$. Denote the inner product of H by $\|\cdot\|$ and the inner product of H' by $\|\cdot\|'$. Since $\|1\| = 1$, we have

$$\|1 \circ \varphi_\lambda \cdot k'_\lambda\|^2 = \|\frac{K_\lambda}{\|K'_\lambda\|'}\|^2 = 1.$$

On the one hand, since $\langle z^\alpha, z^\beta\rangle = 0$ whenever $\alpha \neq \beta$,

$$\|K'_\lambda\|^2 = \sum_{n=0}^{\infty} b_n \|\langle z, \lambda\rangle^n\|^2.$$

On the other hand

$$\|K'_\lambda\|'^2 = \sum_{n=0}^{\infty} b_n |\lambda|^{2n}.$$

Hence

$$\sum_{n=0}^{\infty} b_n \|\langle z, \lambda\rangle^n\|^2 = \sum_{n=0}^{\infty} b_n |\lambda|^{2n}.$$

Taking $\lambda = (r, 0 \cdots, 0)$, we know $\|z_1^n\|^2 = \frac{1}{b_n}$. By [GHX, Proposition 4.1], $\frac{1}{a_n} = \|z_1^n\|^2 = \frac{1}{b_n}$, and hence $K_\lambda = K'_\lambda$, which implies $H = H'$. □

Acknowledgments. The author would like to thank Professor Kunyu Guo for his suggestions and numerous stimulating discussions. The author also want to give his thanks to professor Keiji Izuchi and professor Shuichi Ohno for their hospitalities when the author visited Kyoto.

References

[Ara] J. Arazy, *A Survey of Invariant Hilbert Spaces of Anlytic Functions on Bounded Symmetric Domains*, Contemporaty Mathematics Vol. **185**(1995), 7-65.

[Aro] N. Aronszajn, *Theory of reproducing kernels,* Trans. Amer. Math. Soc. **68**(1950),337-404.

[Arv] W. Arverson, *Subalgebras of C^*-algebras III: Multivariable operator theory,* Acta Math. **181**(1998),159-228.

[GHX] K. Guo, J. Hu and X. Xu, *Toelitz algebras, subnormal tuples and rigidity on reproducing $\mathbb{C}[z_1, \cdots, z_d]$-modules,* J. Funct. Anal. **210**(2004), 214-247.

[Guo] K. Guo, *Defect operators for submodules of H_d^2*, J. Reine Angew. Math. **573**(2004), 181-209.

[Gr] D. Greene, *Free resolutions in multivariable operator theory,* J. Funct. Anal. **200**(2003), 429-450.

[Ru] W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Springer-Berlag, New York, 1980.

Penghui Wang, Department of Mathematics, Fudan University, Shanghai, 200433, P. R. China, E-mail: 031018010@fudan.edu.cn